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SHAKEDOWN OF CREEPING STRUCTURES

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Abstract—The classical approach to shakedown in elastic–plastic structures is extended here to include unlimited time dependent creep in a new way: by way of illustration a simple time-hardening model is used. Shakedown analysis for creep is typically investigated in two steps: an elastic–plastic shakedown analysis based on the classical theorems followed by a correction for creep, for example based on isochronous stress–strain curves for large times with limited creep. This paper provides a different treatment of the problem of estimating creep strains where shakedown could occur by extending the classical shakedown approach to include unlimited creep. New static and kinematic shakedown theorems are established and it is demonstrated that a necessary and sufficient condition for shakedown with mechanical and thermal cyclic loading is the existence of a time-dependent safe residual stress field. Two kinematic theorems are introduced: the first connected with a kinematically admissible cycle of inelastic strain rate and the second with kinematically inadmissible plastic strain rate. In addition all of the results are expressed in terms of generalized variables for structural applications. A few simple examples are examined to verify these results. © 1998 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

At loading around the yield stress and above practically all structural materials give evidence that the formation of inelastic strain is dependent on time, in particular influenced by the rate of loading. Under appropriate conditions, creep and relaxation phenomena are observed—these effects exist even at room temperature, but become much more pronounced at elevated temperature, when accumulation of creep strain is a critical design consideration. When the load is cyclic, the estimation of inelastic strain becomes problematic. This problem has been well recognized even in the earliest literature on shakedown. Many pioneering authors (Bree, 1967; Gokhfeld, 1970; Leckie, 1971; Ponter, 1971; Gokhfeld and Cherniavsky, 1974; O'Donnel and Porowski, 1974; Gokhfeld and Cherniavksy, 1980; Konig, 1987; Rose and White, 1988; Toulios *et al.*, 1991) have considered the possibility of including the effect of time dependent strains (high temperature creep in particular) on shakedown.

Shakedown analysis for creep is typically investigated in two distinct steps: an elastic plastic shakedown analysis based on the classical theorems (Melan, 1938; Koiter, 1956; Koiter, 1960), followed by a correction for creep (Bree, 1967; Ponter, 1971; Gokhfeld and Cherniavsky, 1974; O'Donnel and Porowski, 1974; Gokhfeld and Cherniavsky, 1980). Arguably this can only give a rough estimate of actual shakedown behaviour at best.

In some situations material creep strain accumulation may be limited when secondary creep strain rates are zero. Limited creep appears at stresses close to the yield stress at room temperature in metals, alloys and other structural materials (Rabotnov, 1969). Limited creep is usually irrecoverable when load is removed and is characterized by a relatively short primary phase. These characteristics make it possible to implement an approximate shakedown analysis using isochronous stress–strain curves for large times. However with temperature or stress increasing, creep becomes unlimited. In fact, in most treatments of

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shakedown in the presence of unlimited creep isochronous stress–strain curves are also applied with an age-hardening creep theory at the dwell stage. These curves are then approximated by the elastic–rigid plastic response (O'Donnel and Porowski, 1974; Gokhfeld and Cherniavsky, 1980; Rose and White, 1988; Toulios *et al.*, 1991). Whether or not it is valid as an approximation to use isochronous curves in this manner, it must be emphasized that time dependent creep strains do not occur only at the dwell stage and further that the age-hardening rule gives unsatisfactory results for non-steady loading in general and for cyclic loading in particular.

This paper provides a different approach to the treatment of the problem of the estimation of creep strains under cyclic loads where shakedown could occur. The classical approach to elastic–plastic shakedown of structures is extended to include unlimited creep rather than treating creep as an additional problem; by way of illustration a time-hardening creep model is adopted.

DEFINITION OF THE PROBLEM

In the following the classical results and procedures of Melan (1938) and Koiter (1956, 1960) are extended step-by-step to examine the possibility of including unlimited creep. In this Section, the usual definition of the problem is given in terms of local stress and strain. However in addition this is extended to examine the possibility of using generalised inelastic models (Boyle and Spence, 1983), such as structural beam, plate and shell theories.

We deal here with a structure or solid body, occupying a volume V, under quasistatic cyclic loading. The time history of external actions is periodic in real time. The displacements of all points in the body, as well as their gradients, are assumed to be sufficiently small that changes in geometry are neglected in the equilibrium equations and that the strain–displacement relations are linear.

The total strain rates $\dot{\varepsilon}_{ij}$ are then considered as the sum of four components

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}'_{ij} + \dot{\varepsilon}''_{ij} + \dot{\varepsilon}'''_{ij} + \dot{\varepsilon}^{\theta}_{ij}; \quad i, j = 1, 2, 3 \tag{1}$$

were $\dot{\varepsilon}'_{ij}$, $\dot{\varepsilon}''_{ij}$, $\dot{\varepsilon}'''_{ij}$, $\dot{\varepsilon}^{\theta}_{ij}$ are elastic, plastic, creep and initial (or thermal) strain rates, respectively (Kachanov, 1967).

Elastic strains are given by Hooke's law

$$\varepsilon'_{ij} = A_{ijkl}\sigma_{kl}; \quad i, j, k, l = 1, 2, 3 \tag{2}$$

where σ_{kl} is the stress tensor and A_{ijkl} is a tensor of constant elastic moduli, $A_{ijkl} = A_{jikl} = A_{klij}$. Temperature-dependence of the elastic constants is neglected since this only weakly effects the shakedown condition (Gokhfeld and Cherniavsky, 1980).

Plastic material behaviour is considered to be perfect plasticity with a yield criterion

$$f(\sigma_{ij}) \leqslant K \tag{3}$$

and associated flow rule

$$\dot{\varepsilon}''_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{4}$$

where K is a material constant depending on temperature θ and λ denotes a constant of proportionality such that

$$\lambda = 0 \quad \text{if } f < K \quad \text{or if } f = K \quad \text{and} \quad (\partial f/\partial \sigma_{ij})\dot{\sigma}_{ij} - (\partial K/\partial \theta)\dot{\theta} < 0;$$

$$\lambda \geqslant 0 \quad \text{if } f = K \quad \text{and} \quad (\partial f/\partial \sigma_{ij})\dot{\sigma}_{ij} - (\partial K/\partial \theta)\dot{\theta} = 0$$

Creep strains are given by the time-hardening creep rule derived from a convex homogeneous function $\Phi(\sigma_{ij})$ such that

$$\dot{\varepsilon}_{ij}''' = \frac{\partial \Phi}{\partial \sigma_{ij}}; \quad (\dot{\ }) = \frac{\mathrm{d}()}{\mathrm{d}\tau}; \quad \tau = \int_0^t g[t', \theta(t')]\,\mathrm{d}t' \tag{5}$$

where t is the real time and $g(t, \theta)$ is proportional to the creep strain rates at constant stresses.

The actual structural stress field σ_{ij} and strain field ε_{ij} can be represented by the following decomposition

$$\sigma_{ij} = \sigma_{ij}^{(e)} + \rho_{ij}; \quad \varepsilon_{ij} = \varepsilon_{ij}^{(e)} + \varepsilon_{ijr}$$

where $\varepsilon_{ij}^{(e)}$ and $\sigma_{ij}^{(e)}$ denote the fields which would appear in the structure if the structure was perfectly elastic, while ρ_{ij} and ε_{ijr} are instantaneous residual stress and strain fields.

The residual strain field is further given by

$$\varepsilon_{ijr} = \varepsilon_{ijr}' + \varepsilon_{ij}'' + \varepsilon_{ij}''' \tag{6}$$

where ε_{ijr}' is the residual elastic strain field. According to eqn (2), $\varepsilon_{ijr}' = A_{ijkl}\rho_{kl}$. If a stress field meets the requirement of eqn (3) then it is a safe stress field $\sigma_{ij}^{(s)}$, and the corresponding field variables are denoted $\bar{\rho}_{ij}$, $\bar{\varepsilon}_{ijr}$, $\bar{\varepsilon}_{ijr}'$, $\bar{\varepsilon}_{ij}'''$ ($\bar{\varepsilon}_{ij}'' = 0$ by definition).

Generalized models

This simplified approach to shakedown analysis follows the use of a generalized model of the structure. Generalized models have typically considered structural configurations based on beam, plate or shell theory where the constitutive model is not given in terms of local stress and strain, but instead uses generalized measures of stress and strain, such as beam curvature and moment or shell mid-surface bending and stretching strains and stress resultants (Boyle and Spence, 1983). However this is not a specific restriction on the notion of generalized models—in general these may also be interpreted as substructures (Konig, 1987).

Consider the following, which is analogous to eqn (1):

$$u_m = u_m' + u_m'' + u_m''' + u_m^\theta, \quad m = 1, 2, \ldots$$

where u_m are the generalized displacements, u_m' are the elastic generalized displacements, linearly dependent on the generalized forces Q_m, $u_m' = A_{mn}Q_n$, m, $n = 1, 2, \ldots$, u_m^θ are generalized thermal displacements, u_m'' are the generalized displacements corresponding to perfect plasticity (Konig, 1987) such that

$$\dot{u}_m'' = \Lambda \frac{\partial F}{\partial Q_m}; \quad F(Q_m) \leqslant K';$$

$$\Lambda = 0 \quad \text{if } F < K' \quad \text{or if } F = K' \quad \text{and} \quad \dot{F} - \dot{K}' < 0;$$

$$\Lambda \geqslant 0 \quad \text{if } F = K' \quad \text{and} \quad \dot{F} - \dot{K}' = 0$$

where K' is a parameter of the generalised yield locus for the structure or substructure, and F is a strict convex homogeneous function of the generalised forces,

$$(Q_{m1} - Q_{m2})\dot{u}_{m1}'' > 0 \quad \text{if } F(Q_{m1}) = K' \quad \text{and} \quad F(Q_{m2}) < K' \tag{7}$$

Finally, u_m'' are generalized displacements due to creep—it is straightforward to obtain a time-hardening relationship (Kachanov, 1967; Boyle and Spence, 1983)

$$\dot{u}'''_m = \dot{u}'''_m(Q_n, \theta, t)$$

such that

$$(\dot{u}'''_{m1} - \dot{u}'''_{m2})(Q_{m1} - Q_{m2}) \geqslant 0 \tag{8}$$

due to convexity.

If a structure is considered to be composed of substructures, we will denote external imposed generalized forces of the whole structure by P_m, and generalized displacements and forces of the substructure α ($\alpha = 1, 2, \ldots$) by $u_m^{(\alpha)}$, $Q_m^{(\alpha)}$.

For a substructure (Konig, 1987) we can write

$$\sigma_{ij} = \sigma_{ij}^{(e)} + \sigma_{ij}^{*} + s_{ij}; \quad \sigma_{ij}^{(s)} = \sigma_{ij}^{(e)} + \sigma_{ij}^{*} + \bar{s}_{ij}$$

where σ_{ij}^{*} is a stress field, which would appear in the reference substructure if the substructure was perfectly elastic under the residual generalized forces, and s_{ij} is a corresponding instantaneous residual stress field. The thermo-elastic stress field may be decomposed as follows

$$\sigma_{ij}^{(e)} = \sigma_{ij}^{(ee)} + \rho_{ij}^{\theta}$$

where $\sigma_{ij}^{(ee)}$ is an elastic stress field due to the mechanical loads P_m and ρ_{ij}^{θ} is an elastic stress field due to the temperature gradient.

Then

$$Q_m^{(\alpha)} = Q_m^{(ee)} + Q_m^{(\alpha\theta)} + Q_m^{(\alpha r)}$$

$$\sigma_{ij}^{(ee)}(x_k, t) = Q_m^{(ee)}(t) a_{ij}^m(x_k); \quad \sigma_{ij}^{*}(x_k, t) = Q_m^{(\partial r)}(t) a_{ij}^m(x_k)$$

$$\bar{\sigma}_{ij}^{*}(x_k, t) = \bar{Q}_m^{(\partial, r)}(t) a_{ij}^m(x_k)$$

where x_k are space coordinates, $Q_m^{(ee)}$ is an elastic generalized force field for the substructures due to the mechanical loads P_m, $Q_m^{(\alpha\theta)}$ is a generalized force field for the substructure α due to a temperature stress field and $Q_m^{(\alpha r)}$ is a residual generalized force field for the substructure α under mechanical and thermal unloading.

STATIC SHAKEDOWN THEOREM FOR CREEP

In classical shakedown analysis it is assumed that if a given elastic–plastic structure has already shaken down then the residual stress field will not vary any more (Melan, 1938). For a creeping structure under variable loading the existence of a constant residual stress field is of course impossible. Therefore we will assume here that the necessary condition for shakedown is the existence of a time-dependent safe residual stress field $\bar{\rho}_{ij}$, which, when taken together with the thermo-elastic stresses, is a stress field not violating the yield criterion at any point of a structure at any instant of time.

We now demonstrate that this condition is also a sufficient one:

Theorem. If there exists time-dependent residual stresses $\bar{\rho}_{ij}$ such that when added to the thermo-elastic stresses $\sigma_{ij}^{(e)}$ results in safe stresses $\sigma_{ij}^{(s)}$ at any point of creeping structure at any instant of time

$$\sigma_{ij}^{(s)} = \sigma_{ij}^{(e)} + \bar{\rho}_{ij}; \quad f(\sigma_{ij}^{(s)}) < K \tag{9}$$

then the structure will plastically shake down under periodic loading, that is, its behaviour will become elastic-creeping after some initial cycles, in any load and temperature program within prescribed limits.

To establish this result, we consider the non-negative elastic complementary energy functional of self-equilibrium stresses $\rho_{ij}-\bar{\rho}_{ij}$

$$A(r) = \int_V \frac{1}{2} A_{ijkl}(\rho_{ij}-\bar{\rho}_{ij})(\rho_{kl}-\bar{\rho}_{kl})\,\mathrm{d}V$$

The actual and safe residual fields, and hence A, are time-dependent with the rate of change of A

$$\dot{A}(t) = \int_V A_{ijkl}(\rho_{ij}-\bar{\rho}_{ij})(\dot{\rho}_{kl}-\dot{\bar{\rho}}_{kl})\,\mathrm{d}V \tag{10}$$

Since the residual strain rates $\dot{\varepsilon}_{ijr}$ and $\dot{\bar{\varepsilon}}_{ijr}$ form compatible fields, and since ρ_{ij} and $\bar{\rho}_{ij}$ are both self-equilibrated, it follows from the principle of virtual work that

$$\int_V (\rho_{ij}-\bar{\rho}_{ij})\dot{\varepsilon}_{ijr}\,\mathrm{d}V = 0\,; \quad \int_V (\rho_{ij}-\bar{\rho}_{ij})\dot{\bar{\varepsilon}}_{ijr}\,\mathrm{d}V = 0$$

Therefore, from the equalities given by eqns (6), (9) and (10), we have

$$\dot{A}(t) = -\int_V (\sigma_{ij}-\sigma_{ij}^{(s)})(\dot{\varepsilon}''_{ij}+\dot{\varepsilon}'''_{ij}-\dot{\bar{\varepsilon}}'''_{ij})\,\mathrm{d}V$$

For the convex function Φ

$$(\sigma_{ij}-\sigma_{ij}^{(s)})(\dot{\varepsilon}'''_{ij}-\dot{\bar{\varepsilon}}'''_{ij}) \geqslant 0$$

and since $\sigma_{ij}^{(s)}$ does not violate the yield criterion we have

$$(\sigma_{ij}-\sigma_{ij}^{(s)})\dot{\varepsilon}''_{ij} \geqslant 0$$

Consequently $\dot{A}(t) \leqslant 0$, where the equality holds only in the absence of plastic flow. Since $A \geqslant 0$, the condition $\dot{A} = 0$ must eventually be reached and this condition corresponds to shakedown.

It should be noted that in the above definition of plasticity "time" denotes a loading parameter, in creep it is the real time. As a consequence, time-dependent behaviour of safe stresses should be interpreted as a dependence on the real time. For instance, during instantaneous loading the real time "stops" and the field $\bar{\rho}_{ij}$ is constant since this field is not connected with a possible plastic strain change.

For unlimited creep the above theorem defines the possibility of shakedown. In the case of shakedown stresses, at any point of the structure, moving inside the yield surface, the change in the stress field continues according to the elastic and creep behaviour of the material. Residual stresses $\bar{\rho}_{ij}$ depend entirely on elastic and creep properties of the material and the structure. In order to describe the behaviour of the structure, loaded periodically over a long period of time, for most materials it is possible to assume $g = \text{const}$ or $\lim g(t) = g_\infty$ at $t \to \infty$, where $g_\infty = \text{const}$. In this case a safe stress field is defined as a steady cyclic creep state (Frederick and Armstrong, 1966; Boyle and Spence, 1983).

Now consider the above result in generalized variables. The necessary condition for shakedown in generalized variables can be expressed as

$$F_\alpha(Q_m^{(ee)}+Q_m^{(\alpha\theta)}+\bar{Q}_m^{(\alpha r)}) < K'; \quad \alpha = 1,2,\ldots \tag{11}$$

where the function F_α is connected with appropriate elastic loci and depends on $\bar{s}_{ij}$. The condition described by eqn (11) is also sufficient. Indeed, let us write eqn (10) in the form

$$\dot{A}(t) = \dot{A}_1(t)+\dot{A}_2(t) \tag{12}$$

where

$$\dot{A}_1 = \sum_\alpha \int_{V_\alpha} A_{ijkl}(\sigma_{ij}-\sigma_{ij}^{(s)})(\dot{s}_{kl}-\dot{\bar{s}}_{kl})\,\mathrm{d}V$$

$$\dot{A}_2(t) = \sum_\alpha \int_{V_\alpha} A_{ijkl}(\sigma_{ij}^*-\bar{\sigma}_{ij}^*)(\dot{\sigma}_{kl}^*-\dot{\bar{\sigma}}_{kl}^*)\,\mathrm{d}V+\sum_\alpha \int_{V_\alpha} A_{ijkl}(s_{ij}-\bar{s}_{ij})(\dot{\sigma}_{kl}^*-\dot{\bar{\sigma}}_{kl}^*)\,\mathrm{d}V \tag{13}$$

For clarity in the following discussion, assume the material properties of the substructures are the same.

Now, since

$$A_{ijkl}\dot{s}_{kl} = \dot{\varepsilon}'_{ij} = \dot{\varepsilon}_{ijr}-\dot{\varepsilon}''_{ij}-\dot{\varepsilon}'''_{ij}; \quad A_{ijkl}\dot{\bar{s}}_{kl} = \dot{\bar{\varepsilon}}'_{ij} = \dot{\bar{\varepsilon}}_{ijr}-\dot{\bar{\varepsilon}}'''_{ij}$$

and

$$\sum_\alpha \int_{V_\alpha} (\sigma_{ij}-\sigma_{ij}^{(s)})\dot{\varepsilon}_{ijr}\,\mathrm{d}V = 0; \quad \sum_\alpha \int_{V_\alpha} (\sigma_{ij}-\sigma_{ij}^{(s)})\dot{\bar{\varepsilon}}_{ijr}\,\mathrm{d}V = 0;$$

it follows

$$\dot{A}_1(t) = -\sum_\alpha \int_{V_\alpha} (\sigma_{ij}-\sigma_{ij}^{(s)})(\dot{\varepsilon}''_{ij}+\varepsilon'''_{ij}-\dot{\bar{\varepsilon}}'''_{ij})\,\mathrm{d}V < 0 \tag{14}$$

In eqn (13) the second term is zero since $A_{ijkl}\dot{\sigma}_{kl}^*$, $A_{ijkl}\dot{\bar{\sigma}}_{kl}^*$ are kinematically admissible strain rate fields and s_{ij}, $\bar{s}_{ij}$ are statically admissible self-equilibrated stress fields of each substructure.

Let us define

$$A_{mn}^{(\alpha)} = \int_{V_\alpha} A_{ijkl}a_{ij}^m a_{kl}^n\,\mathrm{d}V$$

then

$$\dot{A}_2(t) = \sum_\alpha A_{mn}^{(\alpha)}(Q_m^{(\alpha r)}-\bar{Q}_m^{(\alpha r)})(\dot{Q}_n^{(\alpha r)}-\dot{\bar{Q}}_n^{(\alpha r)})$$

where

$$A_{mn}^{(\alpha)}\dot{Q}_m^{(\alpha r)} = \dot{u}_n^{(\alpha r)} = \dot{u}'^{(\alpha r)}_n-\dot{u}''_n-\dot{u}'''_n \quad A_{mn}^{(\alpha)}\dot{\bar{Q}}_m^{(\alpha r)} = \dot{\bar{u}}_n^{(\alpha r)} = \dot{\bar{u}}_n^{(\alpha r)}-\dot{\bar{u}}'''_n$$

with $u_m^{(\alpha r)}$ residual generalized displacements and $u_m^{(\alpha r)}$ residual generalized elastic displacements.

From the equilibrium and kinematic conditions of the whole structure we have

$$\sum_{\alpha} Q_m^{(\alpha r)} \dot{u}_m^{(\alpha r)} = 0\,; \quad \sum_{\alpha} (Q_m^{(\alpha r)} - \bar{Q}_m^{(\alpha r)})(\dot{u}_m^{(\alpha r)} - \dot{\bar{u}}_m^{(\alpha r)}) = 0$$

and then

$$\dot{A}_2(t) = -\sum_{\alpha} (Q_m^{(\alpha r)} - \bar{Q}_m^{(\alpha r)})(\dot{u}_m'' + \dot{\bar{u}}_m''' - \dot{u}_m''') = -\sum_{\alpha} (Q_m^{(\alpha)} - \bar{Q}_m^{(\alpha)})(\dot{u}_m'' + \dot{u}_m''' - \dot{\bar{u}}_m''')$$

Now it follows from eqns (7), (8) and (11) that $\dot{A}_2(t) < 0$, and then from eqns (12) and (14) that $\dot{A}(t) < 0$ also. Thus the necessary and sufficient condition for shakedown of the creeping structure is the existence of time dependent residual generalized forces $\bar{Q}_m^{(\alpha r)}$, meeting the condition of eqn (11) for any substructure at any instant of time.

Typically in the formulation of shakedown problems external loads and displacements are given with unknown constants or time-dependent factors are used. In the case of creep the conditions of eqns (9) and (11), as well as in the case of its absence, alternative appropriate conditions (Gokhfeld and Cherniavsky, 1980; Konig, 1987) for the static shakedown theorems are represented as extremum problems with respect to these constants or factors. The presence of creep strains alters the whole structural behaviour, and the shakedown problem becomes the special extremum problem of cyclic creep. This problem is simple to resolve if a cycle of the external actions includes instantaneous overloads, which are required, while creep takes place in the cycle under some prescribed loads. A more complicated situation seems to be the case when creep takes place in a cycle under several combinations of "high" loads. It should be noted that the constitutive equations, eqn (5), only give approximate results for loading conditions which cause large changes (rotations) in the direction of principal stress.

KINEMATIC CONDITIONS OF SHAKEDOWN FOR CREEP

The static theorem proved in the above allows the formulation of several variations of corresponding kinematic theorems. The first uses a kinematically admissible cycle of inelastic strain rates $\dot{\varepsilon}_{ij0}'' + \dot{\varepsilon}_{ij0}'''$ with T as a cycle time. Existence of these strains reflects the stabilization of deformation process under periodic loading (Gokhfeld and Cherniavsky, 1980).

Theorem. The structure will not shakedown, under the thermoelastic stresses $\sigma_{ij}^{(e)}$ corresponding to external actions $Q_m(t)$, $\theta(x_i, t)$ within prescribed limits, if there can be found an admissible cycle of inelastic strain rates $\dot{\varepsilon}_{ij0}'' + \dot{\varepsilon}_{ij0}'''$ such that

$$\int_0^T \mathrm{d}t \int_V (\sigma_{ij} - \sigma_{ij}^{(e)})(\dot{\varepsilon}_{ij0}'' + \dot{\varepsilon}_{ij0}''')\,\mathrm{d}V < \int_0^T (Q_m' - Q_m)\dot{\hat{u}}_m^{(r)}\,\mathrm{d}t - \int_0^T \mathrm{d}t \int_V \varepsilon_{ij}^{\theta} \dot{\hat{\rho}}_{ij}\,\mathrm{d}V \tag{15}$$

where σ_{ij} is the stress field corresponding to the assumed cycle of inelastic strain rate and derived from the load Q_m' and $\dot{\hat{u}}_m^{(r)}$, $\hat{\rho}_{ij}$ correspond to a steady cyclic creep state for given Q_m, θ in the absence of plastic strains.

To establish this result, we assume that the shakedown condition of eqn (9) is valid and a safe residual stress exists, i.e.

$$\hat{\rho}_{ij} = \bar{\rho}_{ij}\,; \quad \dot{\hat{u}}_k^{(r)} = \dot{\bar{u}}_k^{(r)} \tag{16}$$

Then

$$
\begin{aligned}
&\int_0^T \mathrm{d}t \int_V (\sigma_{ij}-\sigma_{ij}^{(e)})(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V \\
&= \int_0^T \mathrm{d}t \int_V (\sigma_{ij}-\sigma_{ij}^{(e)}-\bar{\rho}_{ij})(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V + \int_0^T \mathrm{d}t \int_V \bar{\rho}_{ij}(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V \\
&= \int_0^T \mathrm{d}t \int_V (\sigma_{ij}-\sigma_{ij}^{(s)})(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0}-\dot{\bar{\varepsilon}}'''_{ij})\,\mathrm{d}V + \int_0^T \mathrm{d}t \int_V (\sigma_{ij}-\sigma_{ij}^{(s)})\dot{\bar{\varepsilon}}'''_{ij}\,\mathrm{d}V \\
&\quad + \int_0^T \mathrm{d}t \int_V \bar{\rho}_{ij}(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V
\end{aligned}
$$

where

$$\int_0^T \mathrm{d}t \int_V \bar{\rho}_{ij}(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V = -\int_0^T \mathrm{d}t \int_V \bar{\rho}_{ij}\dot{\varepsilon}'_{ij}\,\mathrm{d}V = \int_0^T \mathrm{d}t \int_V \rho_{ij}\dot{\bar{\varepsilon}}'_{ij}\,\mathrm{d}V;$$

$$\int_V \sigma_{ij}^{(s)}\dot{\bar{\varepsilon}}'''_{ij}\,\mathrm{d}V = \int_V (\sigma_{ij}^{(ee)}+\rho_{ij}^{\theta}+\bar{\rho}_{ij})\dot{\bar{\varepsilon}}'''_{ij}\,\mathrm{d}V = Q_m\dot{\bar{u}}_m^{(r)} + \int_V \bar{\rho}_{ij}\dot{\bar{\varepsilon}}'''_{ij}\,\mathrm{d}V + \int_V \rho_{ij}^{\theta}\dot{\bar{\varepsilon}}'''_{ij}\,\mathrm{d}V;$$

$$\int_V \rho_{ij}^{\theta}\dot{\bar{\varepsilon}}'''_{ij}\,\mathrm{d}V = -\int_V \rho_{ij}^{\theta}\dot{\bar{\varepsilon}}'_{ijr}\,\mathrm{d}V = -\int_V \rho_{ij}^{\theta}A_{ijkl}\dot{\bar{\rho}}_{kl}\,\mathrm{d}V = \int_V \varepsilon_{ij}^{\theta}\dot{\bar{\rho}}_{ij}\,\mathrm{d}V;$$

$$\int_0^T \mathrm{d}t \int_V (\sigma_{ij}-\sigma_{ij}^{(s)})\dot{\bar{\varepsilon}}'''_{ij}\,\mathrm{d}V = \int_0^T (Q'_m-Q_m)\dot{\bar{u}}_k\,\mathrm{d}t - \int_0^T \mathrm{d}t \int_V (\rho_{ij}-\bar{\rho}_{ij})\dot{\bar{\varepsilon}}'_{ij}\,\mathrm{d}V - \int_0^T \mathrm{d}t \int_V \varepsilon_{ij}^{\theta}\dot{\bar{\rho}}_{ij}\,\mathrm{d}V$$

Due to the convexity of the functions f and Φ in eqns (4) and (5)

$$\int_0^T \mathrm{d}t \int_V (\sigma_{ij}-\sigma_{ij}^{(s)})(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0}-\dot{\bar{\varepsilon}}'''_{ij})\,\mathrm{d}V > 0$$

Since the residual stress field $\bar{\rho}_{ij}$ is periodic it follows that

$$\int_0^T \bar{\rho}_{ij}\dot{\bar{\varepsilon}}'_{ij}\,\mathrm{d}t = 0$$

The resulting inequality then can be written as

$$\int_0^T \mathrm{d}t \int_V (\sigma_{ij}-\sigma_{ij}^{(e)})(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V - \int_0^T (Q'_m-Q_m)\dot{\bar{u}}_m^{(r)}\,\mathrm{d}t - \int_0^T \mathrm{d}t \int_V \varepsilon_{ij}^{\theta}\dot{\bar{\rho}}_{ij}\,\mathrm{d}V \geqslant 0$$

This inequality, and the equalities of eqn (16), contradict the assumption in eqn (15). Thus shakedown is impossible if the relation given by eqn (15) is valid.

In terms of generalized variables we write

$$
\begin{aligned}
\int_0^T \mathrm{d}t \int_V \sigma_{ij}^{(e)}(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V &= \int_0^T \mathrm{d}t \int_V \sigma_{ij}^{(ee)}\dot{\varepsilon}_{ij}^{(r)}\,\mathrm{d}V + \int_0^T \mathrm{d}t \int_V \rho_{ij}^{\theta}(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V \\
&= \int_0^T Q_m\dot{u}_{m0}^{(r)}\,\mathrm{d}t + \int_0^T \mathrm{d}t \int_V \varepsilon_{ij}^{\theta}\dot{\rho}_{ij}\,\mathrm{d}V;
\end{aligned}
$$

$$\int_0^T \mathrm{d}t \int_V \sigma_{ij}(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V = \int_0^T Q'_m \dot{u}^{(r)}_{m0}\,\mathrm{d}t + \int_0^T \mathrm{d}t \int_V \rho_{ij}(\dot{\varepsilon}''_{ij0}+\dot{\varepsilon}'''_{ij0})\,\mathrm{d}V$$

The second term is zero, so

$$\int_0^T (Q'_m - Q_m)(\dot{u}^{(r)}_{m0} - \dot{u}^{(r)}_{(m)})\,\mathrm{d}t - \int_0^T \mathrm{d}t \int_V \varepsilon^{\theta}_{ij}(\dot{\rho}_{ij} - \dot{\hat{\rho}}_{ij})\,\mathrm{d}V < 0 \tag{17}$$

The last condition may be used for both the whole structure and for a substructure.

The practical realisation of the conditions given by eqns (15) and (17) supposes the availability of realistic inelastic strain histories (that is, very close to the strain corresponding to the real limit cycle). It is simple to see now that previous simplified kinematically admissible plastic (and creep) strain rates fields, which are usually used in classical shakedown analysis, fail to adequately take into account the influence of creep:

Let $\dot{\varepsilon}''_{ij0}$ be kinematically admissible plastic strain rate. In this case we have the usual strong version of Koiter's Theorem [26], which states that shakedown has not taken place if

$$\int_V (\sigma_{ij} - \sigma^{(e)}_{ij})\dot{\varepsilon}''_{ij0}\,\mathrm{d}V < 0 \tag{18}$$

Indeed, let us assume that the shakedown condition of eqn (9) holds good and a safe residual time-dependent stress field $\bar{\rho}_{ij}$ exists.

$$\int_V (\sigma_{ij} - \sigma^{(e)}_{ij})\dot{\varepsilon}''_{ij0}\,\mathrm{d}V = \int_V (\sigma_{ij} - \sigma^{(s)}_{ij})\dot{\varepsilon}''_{ij0}\,\mathrm{d}V + \int_V \bar{\rho}_{ij}\dot{\varepsilon}''_{ij0}\,\mathrm{d}V$$

where

$$\int_V (\sigma_{ij} - \sigma^{(s)}_{ij})\dot{\varepsilon}''_{ij0}\,\mathrm{d}V \geqslant 0$$

due to convexity.

If $\dot{\varepsilon}''_{ij0}$ are kinematically admissible, and $\bar{\rho}_{ij}$ correspond to the unloaded state, then the last integral vanishes and the resulting inequality contradicts the assumption of eqn (18)!

The requirement of the construction of a realistic strain rate history to estimate the effect of creep complicates the use of the conditions given by eqns (15) and (17). However, this form of the kinematic theorem requires a kinematically inadmissible cycle of plastic strain rates at creep. It is possible to formulate another version of the kinematical condition.

In contrast to the above let σ_{ij} be a stress field corresponding to an assumed arbitrary (not necessarily compatible) cycle of plastic strain rates $\dot{\varepsilon}''_{ij}$ which is on the yield surface.

Theorem. The structure will not shake down if, together with thermoelastic stresses $\sigma^{(e)}_{ij}$ corresponding to external actions given within prescribed limits, there can be found an arbitrary cycle of plastic rates $\dot{\varepsilon}''_{ij0}$ such that

$$\int_0^T \mathrm{d}t \int_V (\sigma_{ij} - \sigma^{(e)}_{ij})\dot{\varepsilon}''_{ij0}\,\mathrm{d}V < \int_0^T \mathrm{d}t \int_V \hat{\rho}_{ij}\dot{\varepsilon}''_{ij0}\,\mathrm{d}V \tag{19}$$

where σ_{ij} are yield surface stresses corresponding to assumed cycle of plastic strain rates and $\hat{\rho}_{ij}$ is a solution corresponding to the steady cyclic creep for the given action in the absence of plastic strain.

To establish this result, assume that the shakedown condition of eqn (9) is valid and that the time-dependent stress field $\hat{\rho}_{ij}$ is a safe residual stress field

$$\hat{\rho}_{ij} = \bar{\rho}_{ij} \tag{20}$$

Then

$$\int_0^T \mathrm{d}t \int_V (\sigma_{ij} - \sigma_{ij}^{(e)})\dot{\varepsilon}''_{ij0} \, \mathrm{d}V - \int_0^T \mathrm{d}t \int_V \bar{\rho}_{ij}\dot{\varepsilon}''_{ij0} \, \mathrm{d}V = \int_0^T \mathrm{d}t \int_V (\sigma_{ij} - \sigma_{ij}^{(s)})\dot{\varepsilon}''_{ij0} \, \mathrm{d}V > 0$$

due to convexity. This inequality together with equality required by eqn (20) contradicts the assumption of eqn (19). Thus shakedown is impossible, if the relation given by eqn (19) holds true.

If during a single cycle plastic flow takes place instantaneously, then it is possible to use a strong version of the above theorem with the condition of inadaptation in the form

$$\int_V (\sigma_{ij} - \sigma_{ij}^{(e)})\dot{\varepsilon}''_{ij0} \, \mathrm{d}V < \int_V \hat{\rho}_{ij}\dot{\varepsilon}''_{ij0} \, \mathrm{d}V \tag{21}$$

where $\dot{\varepsilon}''_{ij0}$ are arbitrary plastic strain rates.

Arbitrary plastic strain rates $\dot{\varepsilon}''_{ij0}$ should be chosen to minimize a locus of external thermo-mechanical loading. In particular, they may be chosen as kinematically admissible plastic strain rates and then the condition of eqn (21) is reduced to the condition of eqn (18), and the effect of creep disappears.

It is straightforward to establish that a structure composed of substructures will not shake down if

$$\int_0^T \sum_\alpha (Q_m'^{(\alpha)} - Q_m^{(\alpha)})\dot{u}''_{m0} \, \mathrm{d}t < \int_0^T \sum_\alpha \hat{Q}_{m0}^{(\alpha r)} \, \dot{u}''_m \, \mathrm{d}t \tag{22}$$

where a cycle of generalized plastic displacement rate fields of substructures $\dot{u}''_{m0}$ is arbitrary (not necessarily compatible) and $Q_m'^{(\alpha)}$ are yield surface generalized forces determined by the plastic displacements rates in accordance with the associated flow rule

$$\dot{u}''_{m0} = \Lambda \, \partial F_\alpha / \partial Q_m^{(\alpha)}$$

The proof of the condition given in eqn (22) is analogous to the above using the convexity condition, eqn (7).

In general, in contrast to the classical approach (Gokhfeld and Cherniavsky, 1980; Konig, 1987) the condition given by eqn (22), as well as that given by eqn (21), does not require kinematically admissible inelastic strain field in substructures. Shakedown conditions arising from the theorems of this section are sufficient ones.

EXAMPLES

Consider the periodic two-step loading of a rectangular section beam by an axial force P and a bending moment M. The beam material is aluminium alloy D16T whose elastic and creep properties at a temperature of 200°C are given in Kuznetsov and Moshkin (1967) and Shorr (1970) as follows:

The modulus of elasticity $E = 60$ GPa, yield stress $\sigma_Y = 330$ MPa.

$$\Phi = \frac{\sigma_e^{n+1}}{(n+1)\sigma^n{}_*}; \quad n = 5.36; \quad \sigma_* = 176.5\ \mathrm{MPa}; \quad g_\infty = 4.6 \cdot 10^{-5}\ \mathrm{h}^{-1}$$

where σ_e is the effective stress, MPa.

The beam has the following dimensions: a half depth is equal to 0.016 m and a half width is equal to 0.01 m.

The first program of loading is given by: $P_1 = 0$ and $M_1 = 588$ Nm at the first step of a cycle. $P_2 = 0$ and $M_1 = 882$ Nm at the second step of a cycle. The first step lasts four hours. Instantaneous cyclic overload, including both loads, is possible at any instant during the cycle.

The steady cyclic creep state for this problem was calculated using a direct method described in Shorr (1964) and Shorr (1970). It is worth noting that this method allows the use of numerical procedures (such as finite element) and can be used for any structure.

To satisfy the kinematic condition, eqn (21), it was assumed that

$$\dot{\varepsilon}_0'' = \mathrm{sgn}(y)C(t)y^\mu; \quad \sigma = \sigma_Y \quad \text{at}\ \dot{\varepsilon}_0'' \geqslant 0 \quad \text{and}\ \sigma = -\sigma_Y \quad \text{at}\ \dot{\varepsilon}_0'' \leqslant 0$$

where y is the dimensionless distance from the middle of depth.

The function $C(t)$, $t \in [0, T]$, may be chosen arbitrarily according to the form of the loading. If we anticipate short-term plastic strains at $t = t_s$, $s = 1, 2, \ldots$, then it would take the form $C \neq 0$ at $t = t_s$ and $C = 0$ at $t \neq t_s$. In the example under consideration $C \neq 0$ at some instant during a cycle. Values of the exponent μ were obtained from the condition of minimum overload.

The locus of safe force combinations corresponding to instantaneous overloads is shown as an interaction diagram in Fig. 1. In this case all loci are symmetric.

The second program of loading assumes: $P_1 = 68.8$ kN and $M_1 = -763$ Nm at the first step of a cycle, $P_2 = 0$ and $M_1 = 882$ Nm at the second step of a cycle. The first step lasts six hours: the cycle is nine hours. Instantaneous cyclic overloads, for both loads, occur at the beginning of each step. The loci of safe forces corresponds to instantaneous overloads, obtained from the condition given by eqn (9), are shown in an interaction diagram in Fig. 2.

It is clear from the interaction diagrams that creep alters the shape, size and location of the shakedown locii.

CONCLUSION

In this paper, the classical approach to shakedown has been extended in a novel way to include unlimited material creep. Unlike most other treatments of the problem, the shakedown and creep conditions are not considered separately. It has been shown that the new static theorem for shakedown at creep implies two kinematic theorems, one connected with a kinematically admissible cycle of inelastic strain rate and the other with a kinematically inadmissible cycle of plastic strain rate.

All the theorems have also been considered in terms of generalized variables.

Although perfect-plasticity and a time-hardening creep model have been used, these results should remain valid for other material models provided they provide convergence of internal stresses in a creeping structure at large times.

These results have been demonstrated on a simple beam problem.

Finally, it is believed that these results can form the basis for a simple and effective inelastic analysis technique for shakedown under creep conditions. Suitable numerical procedures are currently being investigated to implement these new results for generalized models, in particular finite element substructures where the nodal displacement rates and forces on the surface of an assembly of finite elements (considered as a substructure) are taken as the generalized variables. This approach would allow an efficient analysis of large repetitive structures under periodic loading to be considered.

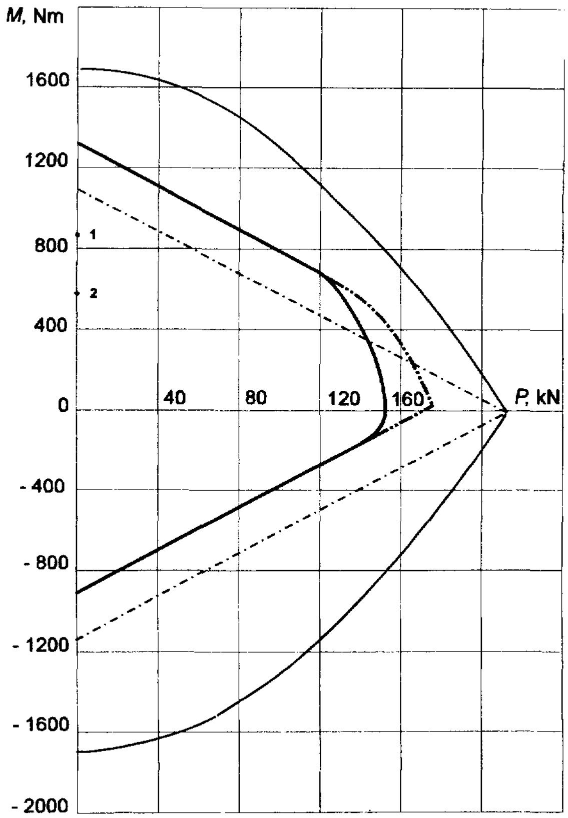

Fig. 1. Load interaction diagram for beam subject to first load case: (points 1 and 2 correspond to the loads at the steps). — Yield locus. -.-.-. Initial elastic locus. — Locus of safe loads for instantaneous overload, eqn (9). -..-..-.. Locus of safe loads for instantaneous overload, upper bound, eqn (19).

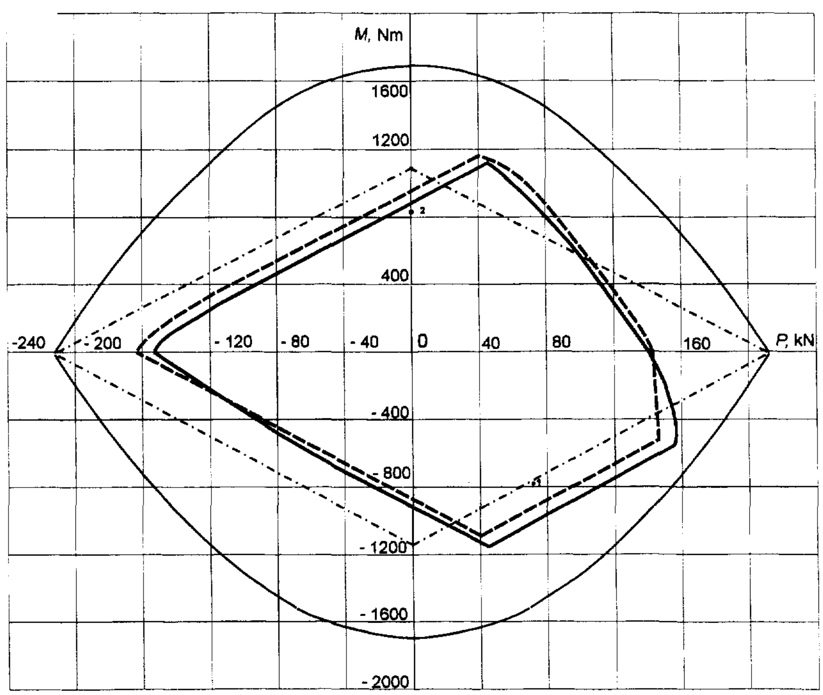

Fig. 2. Load interaction diagram for beam subject to second load case. — Locus of safe loads for instantaneous overload, Step 1, eqn (9). -..-..-.. Locus of safe loads for instantaneous overload, Step 2, eqn (9).

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